J. Larmor Precession: Precession of Spin in a Magnetic Field

General QM: Time-dependent problem is governed by

TDSE $$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

Initial-value problems: $\hat{H}$ does not depend on time (J0)

$\hat{H}\psi_n = E_n \psi_n$; Write $\Psi(x,0) = \sum_i c_i \psi_i$

$$\Psi(x,t) = \sum_i c_i e^{-\frac{iE_i t}{\hbar}} \psi_i$$

Ehrenfest Theorem:

Eqs. similar to classical mechanics, when expectation values are considered

- A charged particle with spin-half
- Not considering its motion in space [not considering orbiting motion, etc], i.e. at rest
- a magnetic field $\vec{B}$

for an electron a proportionality constant

Generally, $\vec{\mu}_s = \frac{q}{m}\vec{S} = \gamma\vec{S}$ (electron: $q = -e$) (J1)

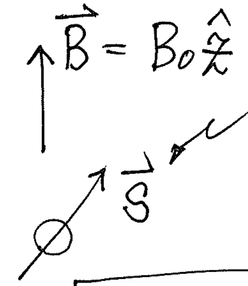
[a magnetic dipole moment is associated with spin AM]

$\vec{\mu}_s$ interacts with $\vec{B}$:

$$H = -\vec{\mu}_s \cdot \vec{B} = -\gamma\vec{S}\cdot\vec{B}$$

Let $\underbrace{\vec{B} = B_0\hat{z}}_{\text{Uniform field (call that direction "z")}}$, $\boxed{H = -\gamma B_0 S_z = -\gamma B_0 \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$ (J2)

This is the Hamiltonian for a charged particle at rest with spin-B field interaction illustration:



there is a quantity called spin AM (thus operators)

charged particle (at rest)

Questions:

- Given state of spin in some arbitrary (general) state at time 0, what is the state at time t?
- How does the state at time t demonstrate the Ehrenfest theorem?

- To handle initial value problem, we need to solve (J2) for H's eigenstates and eigenvalues

Eigenvalue	Eigenstate
$-\gamma B_0 \frac{\hbar}{2} \equiv E_1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha$
$+\gamma B_0 \frac{\hbar}{2} \equiv E_2$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \beta$

(Note: Depending on charge of particle, γ may be negative)

- Consider a general initial spin-half state

$$\psi(0) = \begin{pmatrix} c \\ d \end{pmatrix} = c \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{E_1 \equiv -\gamma B_0 \frac{\hbar}{2}} + d \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{E_2 \equiv \gamma B_0 \frac{\hbar}{2}}$$

- State at time t:

$$\psi(t) = c\, e^{+\frac{i}{\hbar}\gamma B_0 \frac{\hbar}{2} t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + d\, e^{-\frac{i}{\hbar}\gamma B_0 \frac{\hbar}{2} t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (J3)$$

[solved initial value problem]

- $\psi(0)$ is normalized $\Rightarrow$ $(c^* \ d^*)\begin{pmatrix} c \\ d \end{pmatrix} = 1 \Rightarrow |c|^2 + |d|^2 = 1$ (J4)

$\therefore$ Can write $\underbrace{c = \cos\frac{\theta}{2}\,,\ d = \sin\frac{\theta}{2}}_{\text{(J4) is satisfied}}$

- If $\psi(0)$ is normalized, $\psi(t)$ is also normalized. (Ex: check it)

$$\therefore \psi(t) = \cos\frac{\theta}{2}\, e^{i\frac{\gamma B_0}{2}t}\begin{pmatrix}1\\0\end{pmatrix} + \sin\frac{\theta}{2}\, e^{-i\frac{\gamma B_0}{2}t}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\theta}{2}\, e^{i\frac{\gamma B_0}{2}t} \\ \sin\frac{\theta}{2}\, e^{-i\frac{\gamma B_0}{2}t}\end{pmatrix} \qquad (J5)$$

Done! This is how the general initial spin-half state evolves in time in the presence of $B_0\hat{z}$.

- To illustrate a picture of the result and the Ehrenfest theorem, take $\psi(t)$ in Eq.(J5) and calculate expectation values of spin components.

$$\langle \hat{S}_x \rangle = \underbrace{\left(\cos\frac{\theta}{2}\, e^{-i\frac{\gamma B_0}{2}t},\ \sin\frac{\theta}{2}\, e^{i\frac{\gamma B_0 t}{2}} \right)}_{\psi^*} \underbrace{\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\hat{S}_x} \underbrace{\begin{pmatrix} \cos\frac{\theta}{2}\, e^{i\frac{\gamma B_0}{2}\cdot t} \\ \sin\frac{\theta}{2}\, e^{-i\frac{\gamma B_0}{2}\cdot t} \end{pmatrix}}_{\psi}$$

$$= \underbrace{\frac{\hbar}{2}\sin\theta}_{\text{a constant for given } \theta} \cdot \underbrace{\cos(\gamma B_0 t)}_{\text{time varying part}} \qquad \text{(J6)}$$

$$\langle \hat{S}_y \rangle = \left(\cos\frac{\theta}{2}\, e^{-i\frac{\gamma B_0 t}{2}}\ \ \sin\frac{\theta}{2}\, e^{i\frac{\gamma B_0 t}{2}} \right) \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2}\, e^{i\frac{\gamma B_0 t}{2}} \\ \sin\frac{\theta}{2}\, e^{-i\frac{\gamma B_0 t}{2}} \end{pmatrix}$$

$$= -\frac{\hbar}{2}\sin\theta \cdot \underbrace{\sin(\gamma B_0 t)}_{\text{time varying part}} \qquad \text{(J7)}$$

$$\langle \hat{S}_z \rangle = \left(\cos\frac{\theta}{2} e^{-i\gamma \frac{B_0 t}{2}} \quad \sin\frac{\theta}{2} e^{i\gamma \frac{B_0 t}{2}} \right) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} e^{+i\gamma \frac{B_0}{2} t} \\ \sin\frac{\theta}{2} e^{-i\gamma \frac{B_0 t}{2}} \end{pmatrix}$$

$$= \frac{\hbar}{2} \cos\theta \quad \text{(no time varying part)}$$

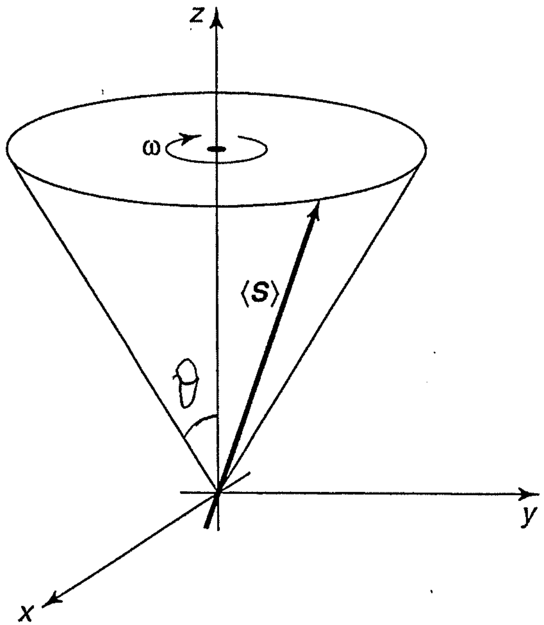
Now, consider $\langle \hat{\vec{S}} \rangle(t) = \langle \hat{S}_x \rangle(t)\, \hat{i} + \langle \hat{S}_y \rangle(t)\, \hat{j} + \langle \hat{S}_z \rangle(t)\, \hat{k}$

Expectation value of spin AM

What does it do in time?

- Look at $\langle \hat{S}_x \rangle(t)$ and $\langle \hat{S}_y \rangle(t)$: Tracing a circle on x-y plane

  $\langle \hat{S}_z \rangle(t)$: z-component doesn't vary with time



- $\langle \hat{\vec{S}} \rangle$ precesses about the field (in z-direction)
- This is what one expects classically for a magnetic dipole moment in a field
- But here this behavior is associated with the expectation value $\langle \hat{\vec{S}} \rangle$ (Ehrenfest theorem)
- θ (in c and d) sets the inclination of $\langle \hat{\vec{S}} \rangle$

- $\omega_{Larmor} = \gamma B_0 = \frac{|e|}{m} B_0$ = Larmor Frequency

If we started off with $\vec{\mu}_s = g \frac{q}{2m} \vec{S}$ (g = g-factor) $g=2$ for electron

then $\omega_{Larmor} = \underbrace{\frac{g|e|}{2m}}_{\text{properties of spin-1/2 particle}} B_0$

ω_{Larmor} can be tuned by external field B_0

<u>Key Point</u>: Spin in a magnetic field gives in characteristic frequency ω_{Larmor}.

[c.f.: 

has a characteristic frequency $\sqrt{\frac{k}{m}} = \omega_0$]

- If we further apply a time-dependence stimulation at some frequency ω, resonance will occur near $\omega \approx \omega_{Larmor}$
- This is the basic idea behind MRI ($\underbrace{\text{nuclear}}_{\text{proton's spin}}$ magnetic resonance)

Theory of time-depending stimulation on a system —
Time-dependent Perturbation Theory (next course)